

Magnetic Correlation Length and Universal Amplitude of the Lattice E_8 Ising Model

M. T. Batchelor* and K. A. Seaton^{†‡}

ANU-MRR-016-97, May 8, 1997

Abstract

The perturbation approach is used to derive the exact correlation length ξ of the dilute A_L lattice models in regimes 1 and 2 for L odd. In regime 2 the A_3 model is the E_8 lattice realisation of the two-dimensional Ising model in a magnetic field h at $T = T_c$. When combined with the singular part f_s of the free energy the result for the A_3 model gives the universal amplitude $f_s \xi^2 = 0.061\,728\dots$ as $h \rightarrow 0$ in precise agreement with the result obtained by Delfino and Mussardo via the form-factor bootstrap approach.

The integrable E_8 quantum field theory of Zamolodchikov [1, 2] is known to be in the same universality class as the two-dimensional Ising model in a magnetic field at $T = T_c$. Moreover, an integrable lattice realisation of the E_8 Ising model is provided by the dilute A_3 model [3, 4], upon which explicit exact and numerical calculations pertaining to the Ising model in a magnetic field can be performed [3-13].

In this letter we present the correlation length of the dilute A_L lattice models in regimes 1 and 2 for L odd, for which the off-critical perturbation is magnetic-like. This includes the magnetic correlation length for $L = 3$, of relevance to the magnetic Ising model at $T = T_c$.

The dilute A_L model is an exactly solvable, restricted solid-on-solid model defined on the square lattice. Each site of the lattice can take one of L possible (height) values, subject to the restriction that neighbouring sites of the lattice either have the same height, or differ by ± 1 . The Boltzmann weights of the allowed height configurations of an elementary face of the lattice are [3, 4]

$$W \begin{pmatrix} a & a \\ a & a \end{pmatrix} = \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)}$$

*Department of Mathematics, School of Mathematical Sciences, The Australian National University, Canberra, ACT 0200, Australia, e-mail murrayb@maths.anu.edu.au

[†]Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, Australia, e-mail k.seaton@latrobe.edu.au

[‡]On leave from School of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia.

$$\begin{aligned}
& - \left(\frac{S(a+1)}{S(a)} \frac{\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + \lambda)} + \frac{S(a-1)}{S(a)} \frac{\vartheta_4(2a\lambda + 5\lambda)}{\vartheta_4(2a\lambda - \lambda)} \right) \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
W \begin{pmatrix} a \pm 1 & a \\ a & a \end{pmatrix} &= W \begin{pmatrix} a & a \\ a & a \pm 1 \end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)} \\
W \begin{pmatrix} a & a \\ a \pm 1 & a \end{pmatrix} &= W \begin{pmatrix} a & a \pm 1 \\ a & a \end{pmatrix} = \left(\frac{S(a \pm 1)}{S(a)} \right)^{1/2} \frac{\vartheta_1(u)\vartheta_4(\pm 2a\lambda - 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)} \\
W \begin{pmatrix} a & a \pm 1 \\ a & a \pm 1 \end{pmatrix} &= W \begin{pmatrix} a \pm 1 & a \pm 1 \\ a & a \end{pmatrix} \\
&= \left(\frac{\vartheta_4(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_4^2(\pm 2a\lambda + \lambda)} \right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}
\end{aligned} \tag{1}$$

$$\begin{aligned}
W \begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} &= \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W \begin{pmatrix} a & a \mp 1 \\ a \pm 1 & a \end{pmatrix} &= - \left(\frac{S(a-1)S(a+1)}{S^2(a)} \right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W \begin{pmatrix} a & a \pm 1 \\ a \pm 1 & a \end{pmatrix} &= \frac{\vartheta_1(3\lambda - u)\vartheta_1(\pm 4a\lambda + 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} + \frac{S(a \pm 1)}{S(a)} \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} \\
&= \frac{\vartheta_1(3\lambda + u)\vartheta_1(\pm 4a\lambda - 4\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)} \\
&\quad + \left(\frac{S(a \mp 1)}{S(a)} \frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)} - \frac{\vartheta_4(\pm 2a\lambda - 5\lambda)}{\vartheta_4(\pm 2a\lambda + \lambda)} \right) \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)}.
\end{aligned}$$

The crossing factors $S(a)$ are defined by

$$S(a) = (-1)^a \frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)} \tag{2}$$

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard elliptic theta functions of nome p

$$\vartheta_1(u) = \vartheta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n}) (1 - p^{2n}) \tag{3}$$

$$\vartheta_4(u) = \vartheta_4(u, p) = \prod_{n=1}^{\infty} (1 - 2p^{2n-1} \cos 2u + p^{4n-2}) (1 - p^{2n}). \tag{4}$$

In the above weights the variable λ and the range of the spectral parameter u are given by $0 < u < 3\lambda$ with

$$\lambda = \frac{s}{r} \pi \tag{5}$$

where $r = 4(L + 1)$ and $s = L$ in regime 1 and $s = L + 2$ in regime 2.¹ The magnetic Ising point occurs in regime 2 with $\lambda = \frac{5\pi}{16}$.

The row transfer matrix of the dilute A models is defined on a periodic strip of width N as

$$T_{\{a\}}^{\{b\}} = \prod_{j=1}^N W \begin{pmatrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{pmatrix} \quad (6)$$

where $\{a\}$ is an admissible path of heights and $a_{N+1} = a_1$, $b_{N+1} = b_1$. For convenience we take N even.

The eigenvalues of the transfer matrix are [6, 14, 15]

$$\begin{aligned} \Lambda(u) &= \omega \left[\frac{\vartheta_1(2\lambda - u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j + \lambda)}{\vartheta_1(u - u_j - \lambda)} \\ &+ \left[\frac{\vartheta_1(u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j) \vartheta_1(u - u_j - 3\lambda)}{\vartheta_1(u - u_j - \lambda) \vartheta_1(u - u_j - 2\lambda)} \\ &+ \omega^{-1} \left[\frac{\vartheta_1(u) \vartheta_1(\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j - 4\lambda)}{\vartheta_1(u - u_j - 2\lambda)} \end{aligned} \quad (7)$$

where the N roots u_j are given by the Bethe equations

$$\omega \left[\frac{\vartheta_1(\lambda - u_j)}{\vartheta_1(\lambda + u_j)} \right]^N = - \prod_{k=1}^N \frac{\vartheta_1(u_j - u_k - 2\lambda) \vartheta_1(u_j - u_k + \lambda)}{\vartheta_1(u_j - u_k + 2\lambda) \vartheta_1(u_j - u_k - \lambda)} \quad (8)$$

and $\omega = \exp(i\pi\ell/(L + 1))$ for $\ell = 1, \dots, L$.

There are several methods at hand to calculate the correlation length. Here we apply the perturbative approach initiated by Baxter [16, 17]. For L odd this involves perturbing away from the strong magnetic field limit at $p = 1$. We thus introduce the variables

$$w = e^{-2\pi u/\epsilon} \quad \text{and} \quad x = e^{-\pi^2/r\epsilon} \quad (9)$$

conjugate to the nome $p = e^{-\epsilon}$. The relevant conjugate modulus transformations are

$$\vartheta_1(u, p) = \left(\frac{\pi}{\epsilon} \right)^{1/2} e^{-(u-\pi/2)^2/\epsilon} E(w, q^2) \quad (10)$$

$$\vartheta_4(u, p) = \left(\frac{\pi}{\epsilon} \right)^{1/2} e^{-(u-\pi/2)^2/\epsilon} E(-w, q^2) \quad (11)$$

where $q = e^{-\pi^2/\epsilon}$ and

$$E(z, p) = \prod_{n=1}^{\infty} (1 - p^{n-1}z)(1 - p^n z^{-1})(1 - p^n). \quad (12)$$

In the ordered limit ($p \rightarrow 1$ with u/ϵ fixed) the Boltzmann weights for L odd reduce to

$$W \begin{pmatrix} d & c \\ a & b \end{pmatrix} \sim w^{H(d,a,b)} \delta_{a,c}. \quad (13)$$

¹The model has other regimes, but they are not of interest here.

The function $H(d, a, b)$ is given explicitly in [5], being required for the calculation of the local height probabilities. In this limit the row transfer matrix eigenspectra breaks up into a number of bands labelled by integer powers of w . In regime 1 there are $\frac{1}{2}(L+1)$ ground states and in regime 2 there are $\frac{1}{2}(L-1)$ ground states, each with eigenvalue $\Lambda_0 = 1$. The bands of excitations are relevant to the calculation of the correlation length.

The number of states in the w band is $\frac{1}{2}(L-1)N$ in regime 1 and $\frac{1}{2}(L-3)N$ in regime 2. These correspond to introducing in all but one of the ground state paths $\{a\}$ a single non-ground state height, in any position. In particular, note that there are *no* excitations in the w band for $L = 3$ in regime 2. Thus for the magnetic Ising model we must consider excitations in the w^2 band. These are harder to count, arising from a variety of both single and multiple deviations from ground state paths. However, we observe numerically that (apart from when $N = 2$) there are $4N$ states in the w^2 band.

We associate a given value of ℓ with each eigenvalue by numerically comparing the eigenspectrum at criticality ($p = 0$) with the eigenspectrum of the corresponding $O(n)$ loop model [18] for finite N .² Each eigenvalue can then be tracked to the ordered limit. The band of largest eigenvalues is seen to have the values $\ell = 1, \dots, \frac{1}{2}(L+1)$ in regime 1 and $\ell = 1, \dots, \frac{1}{2}(L-1)$ in regime 2.

Setting $w_j = e^{-2\pi u_j/\epsilon}$, the eigenvalues (7) can be written

$$\begin{aligned} \Lambda(w) = & \omega \left[\frac{E(x^{4s}/w, x^{2r}) E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j^{1-2s/r} \frac{E(x^{2s}w/w_j, x^{2r})}{E(x^{2s}w_j/w, x^{2r})} \\ & + \left[\frac{x^{2s}}{w} \frac{E(w, x^{2r}) E(x^{6s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j \frac{E(w/w_j, x^{2r}) E(x^{6s}w_j/w, x^{2r})}{E(x^{2s}w_j/w, x^{2r}) E(x^{4s}w_j/w, x^{2r})} \\ & + \omega^{-1} \left[x^{2s} \frac{E(w, x^{2r}) E(x^{2s}/w, x^{2r})}{E(x^{4s}, x^{2r}) E(x^{6s}, x^{2r})} \right]^N \prod_{j=1}^N w_j^{2s/r} \frac{E(x^{8s}w_j/w, x^{2r})}{E(x^{4s}w_j/w, x^{2r})}. \end{aligned} \quad (14)$$

The Bethe equations (8) are now

$$\omega \left[w_j \frac{E(x^{2s}/w_j, x^{2r})}{E(x^{2s}w_j, x^{2r})} \right]^N = - \prod_{k=1}^N w_k^{2s/r} \frac{E(x^{2s}w_j/w_k, x^{2r}) E(x^{4s}w_k/w_j, x^{2r})}{E(x^{2s}w_k/w_j, x^{2r}) E(x^{4s}w_j/w_k, x^{2r})}. \quad (15)$$

The calculation of the largest eigenvalue proceeds from the $x \rightarrow 0$ limit with w fixed in a similar manner to that for the eight-vertex [16] and CSOS [17] models. Each of the degenerate ground states has a different root distribution $\{w_j\}$ on the unit circle, depending on ℓ . Defining the free energy per site as $f = N^{-1} \log \Lambda_0$ our final result is

$$f = 4 \sum_{k=1}^{\infty} \frac{\cosh[(5\lambda - \pi)\pi k/\epsilon] \cosh(\pi \lambda k/\epsilon) \sinh(\pi u k/\epsilon) \sinh[(3\lambda - u)\pi k/\epsilon]}{k \sinh(\pi^2 k/\epsilon) \cosh(3\pi \lambda k/\epsilon)} \quad (16)$$

in agreement with the previous calculations via the inversion relation method [3, 4, 5].

²Strictly speaking we compare with the eigenspectrum of the corresponding vertex model with seam ω .

In regime 1, the leading eigenvalue in the w band has $\ell = \frac{1}{2}(L + 1) + 1$. The root distribution has $N - 1$ roots on the unit circle and a 1-string excitation located exactly at $w_N = -x^r$. Applying perturbative arguments along the lines of [17] yields the leading excitation in the w band to be

$$\frac{\Lambda_1}{\Lambda_0} = w \frac{E(-x^{2s}/w, x^{12s}) E(-x^{4s}/w, x^{12s})}{E(-x^{2s} w, x^{12s}) E(-x^{4s} w, x^{12s})}. \quad (17)$$

At the isotropic point $w = x^{3s}$ this reduces to

$$\frac{\Lambda_1}{\Lambda_0} = x^s \frac{E^2(-x^s, x^{12s})}{E^2(-x^{5s}, x^{12s})} = \left[\frac{\vartheta_4(\frac{\pi}{12}, p^{\pi/6\lambda})}{\vartheta_4(\frac{5\pi}{12}, p^{\pi/6\lambda})} \right]^2. \quad (18)$$

For $L = 3$ in regime 2 extensive numerical investigations of the Bethe equations have lead to a convincing conjecture for the thermodynamically significant strings [6, 9]. We find that the leading excitation in the w^2 band is a 2-string with $\ell = 2$. However, the state is originally a 1-string for small p . Such behaviour has been discussed in [9]. Tracking this state with increasing p reveals that the 2-string is exactly located at $-x^{\pm 11}$ in the limit $p = 1$. There are finite-size deviations away from this position for small N and $0 < p < 1$. The location we find for this string is in accord with the previous numerical work [6, 9]. Applying the perturbation arguments in this case yields the leading excitation in the w^2 band for $L = 3$ to be

$$\frac{\Lambda_2}{\Lambda_0} = w^2 \frac{E(-x/w, x^{60}) E(-x^{11}/w, x^{60}) E(-x^{31} w, x^{60}) E(-x^{41} w, x^{60})}{E(-x w, x^{60}) E(-x^{11} w, x^{60}) E(-x^{31}/w, x^{60}) E(-x^{41}/w, x^{60})}. \quad (19)$$

At the isotropic point $w = x^{15}$ this reduces to

$$\frac{\Lambda_2}{\Lambda_0} = x^{28} \frac{E^2(-x^4, x^{60}) E^2(-x^{14}, x^{60})}{E^2(-x^{16}, x^{60}) E^2(-x^{26}, x^{60})} = \left[\frac{\vartheta_4(\frac{\pi}{15}, p^{8/15}) \vartheta_4(\frac{7\pi}{30}, p^{8/15})}{\vartheta_4(\frac{4\pi}{15}, p^{8/15}) \vartheta_4(\frac{13\pi}{30}, p^{8/15})} \right]^2. \quad (20)$$

The correlation length ξ can be obtained either by integrating over the relevant band of eigenvalues or via the leading eigenvalue in the band at the isotropic point (see, e.g., [17]). Doing the latter we have

$$\xi^{-1} = -\log \frac{\Lambda}{\Lambda_0} \quad (21)$$

where Λ is the relevant leading eigenvalue. Our final results are thus

$$\xi^{-1} = 2 \log \left[\frac{\vartheta_4(\frac{5\pi}{12}, p^{\pi/6\lambda})}{\vartheta_4(\frac{\pi}{12}, p^{\pi/6\lambda})} \right] \quad (22)$$

for L odd in regime 1, with

$$\xi^{-1} = 2 \log \left[\frac{\vartheta_4(\frac{4\pi}{15}, p^{8/15}) \vartheta_4(\frac{13\pi}{30}, p^{8/15})}{\vartheta_4(\frac{\pi}{15}, p^{8/15}) \vartheta_4(\frac{7\pi}{30}, p^{8/15})} \right] \quad (23)$$

for $L = 3$ in regime 2.

The derivation of the correlation length for $L \neq 3$ in regime 2 is complicated. In this regime the leading excitation in the w band has $\ell = \frac{1}{2}(L - 1) + 1$ and, like

the leading 2-string in the w^2 band for $L = 3$, it begins life for small N and $p \simeq 0$ as a 1-string. We have not pursued this further. Nevertheless we have numerically observed that the final result (17) also applies to the leading w band excitation in regime 2. We thus believe that the correlation length (22) and the corresponding exponents below also hold in regime 2 for $L \neq 3$.

It follows from (22) that the correlation length diverges at criticality as

$$\xi \sim \frac{1}{4\sqrt{3}} p^{-\nu_h} \quad \text{as } p \rightarrow 0 \quad (24)$$

where the correlation length exponent ν_h is given by

$$\nu_h = \frac{r}{6s} = \begin{cases} \frac{2(L+1)}{3L} & \text{regime 1} \\ \frac{2(L+1)}{3(L+2)} & \text{regime 2} . \end{cases} \quad (25)$$

The correlation length exponents are seen to satisfy the general scaling relation $2\nu_h = 1 + 1/\delta$, which follows from the relation

$$f_s \xi^2 \sim \text{constant} \quad (26)$$

where $f_s \sim p^{1+1/\delta}$ is the singular part of the bulk free energy and the exponents δ are those following from the singular behaviour of (16) [3, 4, 5].³

The magnetic Ising case at $\lambda = \frac{5\pi}{16}$ is of particular interest. From (16) we find

$$f_s \sim 4\sqrt{3} \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{30}} p^{16/15} \quad \text{as } p \rightarrow 0. \quad (27)$$

On the other hand, from (23) we have

$$\xi \sim \frac{1}{8\sqrt{3} \sin \frac{\pi}{5}} p^{-8/15} \quad \text{as } p \rightarrow 0. \quad (28)$$

Combining these results gives the universal magnetic Ising amplitude

$$f_s \xi^2 = \frac{1}{16\sqrt{3} \sin \frac{\pi}{5} \cos \frac{\pi}{30}} = 0.061\,728\,589\dots \quad \text{as } p \rightarrow 0. \quad (29)$$

This is in precise agreement with the field-theoretic result obtained recently by Delfino and Mussardo, starting from Zamolodchikov's S -matrix and using the form-factor bootstrap approach [19, 20]. Full details of our calculations will be given elsewhere.

It is a pleasure to thank John Cardy and Ole Warnaar for some helpful remarks. The work of KAS has been facilitated by a Commonwealth Staff Development Fund grant, administered by the Academic Development Unit of La Trobe University. The work of MTB has been supported by the Australian Research Council.

³The same correlation length exponents should hold for L even, for which the integrable perturbation is thermal-like. The scaling relation is now $2\nu_t = 2 - \alpha$, where ν_t and α are as given in (25) and [3, 4, 5], respectively. In particular, (25) gives the Ising value $\nu_t = 1$ for $L = 2$ in regime 1, as expected.

References

- [1] A. B. Zamolodchikov, Adv. Stud. Pure Math. 19 (1989) 641.
- [2] A. B. Zamolodchikov, Int. J. Mod. Phys. A 4 (1989) 2371.
- [3] S. O. Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. 69 (1992) 710.
- [4] S. O. Warnaar, B. Nienhuis and K. A. Seaton, Int. J. Mod. Phys. B 7 (1993) 3727.
- [5] S. O. Warnaar, P. A. Pearce, K. A. Seaton and B. Nienhuis, J. Stat. Phys. 74 (1994) 469.
- [6] V. V. Bazhanov, B. Nienhuis and S. O. Warnaar, Phys. Lett. B 322 (1994) 198.
- [7] S. O. Warnaar and P. A. Pearce, J. Phys. A 27 (1994) L891.
- [8] D. L. O'Brien and P. A. Pearce, J. Phys. A 28 (1995) 4891.
- [9] U. Grimm and B. Nienhuis, Phys. Rev. E 55 (1997) 5011.
- [10] B. M. McCoy and W. P. Orrick, hep-th/9611071.
- [11] M. T. Batchelor, V. Fridkin and Y. K. Zhou, J. Phys. A 29 (1996) L61.
- [12] Y. K. Zhou and M. T. Batchelor, Nucl. Phys. B 485 (1997) 646.
- [13] K. A. Seaton and L. C. Scott, "q-Trinomial coefficients and the dilute A model", La Trobe University Technical Report No. 11, 1997.
- [14] Y. K. Zhou, P. A. Pearce and U. Grimm, Physica A 222 (1995) 261.
- [15] Y. K. Zhou, Int. J. Mod. Phys. B 10 (1996) 3481.
- [16] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic Press, London, 1982).
- [17] P. A. Pearce and M. T. Batchelor, J. Stat. Phys. 60 (1990) 77.
- [18] S. O. Warnaar and B. Nienhuis, J. Phys. A 26 (1993) 2301.
- [19] V. A. Fateev, Phys. Lett. B 324 (1994) 45.
- [20] G. Delfino and G. Mussardo, Nucl. Phys. B 455 (1995) 724.